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Models for SU(3) in terms of so(n, 2) and so*($2n$) algebras

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Abstract. Mathematical and physical models for SU(3) are given in terms of unirreps of the so(n, 2) and so*($2n$) algebras. Vector coherent state representations are given for all the discrete series unirreps of SO(n, 2) and SO*($2n$). Traceless bosons are shown to arise naturally in these representations. A new and elegant realisation of the so*(8) $\sim$ so(6, 2) model for SU(3) is given and it is shown how to perform calculations in the so*(8) model using the powerful vector coherent state representations formalism.

1. Introduction

In recent times, the term 'model' has come to acquire rather specific meanings both in physics and in the representation theory of Lie groups. A model in physics is particularly useful when it is characterised by a dynamical group or, equivalently, when it is expressible in terms of a spectrum generating algebra which is the Lie algebra of a dynamical group. Given a model Hamiltonian, it was traditional to seek a symmetry group that would explain its degeneracies. However, it has recently become apparent that it is more generally useful to seek a dynamical group that will at once explain the characteristics of a whole spectrum. The dynamical group should ideally, and usually does, include symmetry groups as subgroups. More important, however, is the fact that a realisation of the model is provided by an irreducible unitary representation of its dynamical group. Thus the carrier space for the unirrep of the dynamical group is a Hilbert space for the model. In the theory of nuclear collective motion, for example, the symplectic group Sp(3, $\mathfrak{R}$) is the dynamical group for the microscopic collective model and U(6) is the dynamical group for the interacting boson model.

In the theory of group representations of compact Lie groups, the term model has recently acquired a closely related meaning. Given a compact Lie group G, a model for this Lie group is defined as 'a realisation of a representation of G which consists of a direct sum of irreducible representations (irreps), containing exactly one representative from every equivalence class of irreps of G' (Bracken and MacGibbon 1984, Bernšteĭn *et al* 1975). Again it is very useful if such a model carries an irrep of a dynamic group H which must now necessarily be non-compact. Usually it will be required that G be a subgroup of H but in some instances, one of which is given in this paper, we may dispense with such a requirement. In the latter case, the Lie algebra of G will belong to the enveloping algebra of H.

For example, the group product SU(2)[HW(2)], with HW(2) a two-dimensional Heisenberg-Weyl group, has been used by Schwinger to study the tensor structure of

SU(2). More recently, a unirrep of the non-compact semisimple group SO(6, 2) was identified as a model for SU(3) and used to study the tensor structure of SU(3). Realisations of the SO(6, 2) model were given by Biedenharn and Flath (1984), Bracken and MacGibbon (1984) and Bracken (1984). Naturally one would like a model realisation that has useful properties like unitarity, economy in terms of the number of variables required for its realisation, ease of computation in the model space and, if possible, the realisation should naturally express itself in a canonical (Cartan) basis. All of the above-mentioned realisations of the SO(6, 2) model suffer in one or many of these aspects which is one of the motivations for the present work.

A difficulty with SO(6, 2) as a dynamical group for SU(3) is the fact that SU(3) is not embedded in SO(6, 2) in an obvious manner. We shall show that, in contrast, the locally isomorphic SO*(8) group does not have this problem. For, whereas the maximal compact subgroup of SO(6, 2) is SO(6)$\otimes$SO(2), that of SO*(8) is U(4). Thus SU(3) is canonically embedded in U(4) and hence in SO*(8). The identification of the SO*(8) structure leads to some valuable new insights, among which is the recognition that the so*(8) algebra naturally contains the fundamental Wigner operators as defined by Biedenharn and Louck (cf Biedenharn and Flath 1984).

We give a realisation of the so*(8) $\sim$ so(6, 2) algebra which is naturally expressed in a Cartan basis, is unitary and quadratic in two Bargmann 4-vectors. It is shown that the non-unitary realisation of so(6, 2) expressed in terms of traceless bosons by Bracken and MacGibbon (1984) is one of a set of coherent state representations of this algebra given by the recent vector coherent state representation theory of Rowe (1984) and Rowe *et al* (1985a). It is also isomorphic to one of an equivalent set of partially coherent state representations of Deenen and Quesne (1984). Furthermore, it is shown how these coherent state representations can be utilised to calculate reduced matrix elements of the so*(8) and so(6, 2) algebras.

A second motivation for the present analysis is the possibility of utilising the concept of an SU(3) model in the development of the microscopic theory of nuclear collective motion. As mentioned above, the non-compact symplectic group Sp(3, $\mathfrak{R}$) is a dynamical group for the microscopic collective model. The relevant Sp(3, $\mathfrak{R}$) unirreps are lowest weight representations, each having a lowest weight state that is simultaneously a lowest weight state for a unirrep of U(3), the maximal compact subgroup of Sp(3, $\mathfrak{R}$). Now, an Sp(3, $\mathfrak{R}$) lowest weight state can be identified with an intrinsic state of the collective model. The rotations and high frequency vibrations of this intrinsic state are described by the Sp(3, $\mathfrak{R}$) collective model. However, the experimental data seem to suggest the occurrence of additional low frequency (beta and gamma) vibrational degrees of freedom that are not embraced by the Sp(3, $\mathfrak{R}$) model. It is of major interest therefore to seek another dynamical group, in some sense complementary to Sp(3, $\mathfrak{R}$), appropriate for those low frequency collective vibrations. Such a group should connect different Sp(3, $\mathfrak{R}$) lowest weight states. The discovery of such a dynamical group and its subsequent realisation in terms of both collective and single-particle variables would evidently represent a major advance in nuclear theory. Among other things, it could give the long sought microscopic interpretation of the very successful phenomenological interacting boson model. Evidently, the SO*(8) $\sim$ SO(6, 2) groups are candidates for such a dynamical group. But it is also possible that a microscopic realisation of specific unirreps of the so(A, 2) algebra, which is thoroughly reviewed in § 5, could be used to generate three-rowed representations of SO(A) and, by complementarity (Moshinsky and Quesne 1970, 1971), SO(A)$\otimes$Sp(3, $\mathfrak{R}$) lowest weight states for the symplectic model of A nucleon collective motions.

2. Models for the symmetrical representations of SU(3)

Before introducing the SO*(8) model spanning all generic $\{h_1h_2\}$ SU(3) unirreps, we will discuss simpler models for the space of all symmetrical unirreps of SU(3) labelled by the characters $\{h_10\}$.

A possible dynamical group for the one-rowed representations of SU(3) has been found to be the semi-direct product group U(3)[HW(3)] (Haskell and Wybourne 1973) with the normal subgroup HW(3) a three-dimensional Heisenberg–Weyl group. If we relax the requirement that the dynamical group contains SU(3) but still retains a sufficient structure to classify all states, we obtain the minimal group

$$(\mathrm{U}(1)\otimes\mathrm{SO}(3))[\mathrm{HW}(3)] \tag{2.1}$$

where U(1)) is the one-dimensional group generated by the boson number operator. One-rowed representations are then uniquely labelled by the U(1) label h_1 and a basis

$$|\{h_10\}LM\rangle \tag{2.2}$$

is uniquely labelled by the $\mathrm{SO}(3)\supset\mathrm{SO}(2)$ labels. Such a simplification is allowed as the $\mathrm{SU}(3)\downarrow\mathrm{SO}(3)$ branching is multiplicity free for SU(3) symmetrical unirreps.

The fundamental representation of this group is carried by the space of Bargmann polynomials in a three-dimensional vector g. Bases for the Lie algebras acting on the Bargmann polynomials are given respectively by

$$\begin{aligned} &\mathrm{HW}(3){:} && \{g_i, \partial/\partial g_j, \delta_{ij}\},\\ &\mathrm{U}(1){:} && \{N=g_i\partial/\partial g_i\},\\ &\mathrm{SO}(3){:} && \{L_i=-\mathrm{i}\varepsilon_{ijk}g_j\partial/\partial g_k\}. \end{aligned} \tag{2.3}$$

The Lie algebra for U(3)

$$\mathrm{U}(3){:}\qquad \{C_{ij}=g_i\partial/\partial g_j\} \tag{2.4}$$

is seen to be embedded in the enveloping algebra of $(\mathrm{U}(1)\otimes\mathrm{SO}(3))[\mathrm{HW}(3)]$.

It is clear that the polynomials of degree h_1 in the Bargmann variables g_i carry the unirrep $\{h_10\}$ of SU(3). To construct the corresponding basis, it is convenient to exploit the local isomorphism $\mathrm{so}(3)\simeq\mathrm{su}(2)$. Under this isomorphism, the Bargmann variables become components of an SU(2) symmetrical tensor of rank $\{2\}$ or a U(2) tensor of rank $\{20\}$. It follows that the Lie algebra of the group product $(\mathrm{U}(1)\otimes\mathrm{SO}(3))[\mathrm{HW}(3)]$ is isomorphic to a u(2)-boson algebra U(2)[HW(3)] for which a canonical basis has already been constructed (Le Blanc and Rowe 1985a, b, Quesne 1981). The U(2) content of this irreducible representation of the u(2)-boson algebra is given by the set of U(2) unirreps characterised by partitions $\{n_1n_2\}$ with n_1,n_2 even integers and $n_1\geqslant n_2$. Using the isomorphism, we can make the identification

$$\begin{array}{l} |\ \{h_10\}\qquad L\qquad M\ \rangle \equiv |\{n_1n_2\}\qquad \nu\ \rangle \\ \ \ \mathrm{SU}(3)\supset\mathrm{SO}(3)\supset\mathrm{SO}(2)\qquad \mathrm{U}(2)\ \supset\mathrm{U}(1) \end{array} \tag{2.5a}$$

with

$$h_1=\frac{n_1+n_2}{2},\qquad L=\frac{n_1-n_2}{2},\qquad M=\nu. \tag{2.5b}$$

It is easily verified that this algebra effectively generates a space isomorphic to the direct sum of all symmetrical $\mathrm{SU}(3)\supset\mathrm{SO}(3)$ unirrep spaces (cf figure 2).

Observe that the Bargmann variables and the differentials in them are Wigner operators, i.e. if in a spherical basis we define

$$T^{\{10\}}_{1m} = g_m, \qquad T^{\{11\}}_{1m} = (-1)^m \partial/\partial g_{-m}, \tag{2.6}$$

then

$$\begin{aligned} T^{\{10\}}_{1m}|\{h_1 0\}LM\rangle &= \sum_{L'} c_{L'}|\{h_1+1, 0\}L'M+m\rangle, \\ T^{\{11\}}_{1m}|\{h_1 0\}LM\rangle &= \sum_{L'} d_{L'}|\{h_1-1, 0\}L'M+m\rangle, \end{aligned} \tag{2.7a}$$

with

$$L' = L \pm 1, \tag{2.7b}$$

and

$$\begin{aligned} c_{L'} &= \langle LM; 1m|L'M+m\rangle\langle\{h_1+1, 0\}L'\| T^{\{10\}}_1 \|\{h_1 0\}L\rangle, \\ d_{L'} &= \langle LM; 1m|L'M+m\rangle\langle\{h_1-1, 0\}L'\| T^{\{11\}}_1 \|\{h_1 0\}L\rangle. \end{aligned} \tag{2.7c}$$

The SO(3)-reduced matrix elements in (2.7c) are given immediately by the known SU(2)-reduced matrix elements of the u(2)-boson algebra (Le Blanc and Rowe 1985a, Quesne 1981)

$$\langle\{h_1+1, 0\}L'\| T^{\{10\}}_1 \|\{h_1\ 0\}L\rangle = (\{n_1'n_2'\}\|a^+\|\{n_1 n_2\}) \tag{2.8a}$$

with

$$n_1' = h_1+1+L', \qquad n_2' = h_1+1-L'. \tag{2.8b}$$

We find

$$\begin{aligned} \langle\{h_1+1, 0\}L+1\| T^{\{10\}}_1 \|\{h_1 0\}L\rangle &= \left(\frac{(h_1+L+3)(L+1)}{(2L+3)}\right)^{1/2}, \\ \langle\{h_1+1, 0\}L-1\| T^{\{10\}}_1 \|\{h_1 0\}L\rangle &= \left(\frac{L(h_1-L+2)}{(2L-1)}\right)^{1/2}. \end{aligned} \tag{2.9}$$

A knowledge of these matrix elements enables one to easily calculate matrix elements of the other Wigner operator $T^{\{11\}} = (T^{\{10\}})^+$ and of higher rank tensors in an SO(3) basis. For example, the SU(3) algebra is given by (Le Blanc and Rowe 1985b, Haskell and Wybourne 1973)

$$T^{\{21\}}_{lm} = (-1)^l \sqrt{2}[T^{\{10\}}_1 \times T^{\{11\}}_1]_{lm}, \qquad l = 1, 2. \tag{2.10}$$

We find

$$\begin{aligned} \langle\{h_1, 0\}L'\| T^{\{21\}}_1 \|\{h_1 0\}L\rangle &= \delta_{LL'}[L(L+1)]^{1/2}, \\ \langle\{h_1, 0\}L+2\| T^{\{21\}}_2 \|\{h_1 0\}L\rangle &= -\left(\frac{2(L+1)(L+2)(h_1-L)(h_1+L+3)}{(2L+3)(2L+5)}\right)^{1/2}, \\ \langle\{h_1, 0\}L\| T^{\{21\}}_2 \|\{h_1 0\}L\rangle &= -(2h_1+3)\left(\frac{L(L+1)}{3(2L-1)(2L+3)}\right)^{1/2}. \end{aligned} \tag{2.11}$$

The group (U(1)⊗SO(3))[HW(3)] can be regarded as a contraction of SO(3, 2). Indeed, it is known that SO(3, 2) is locally isomorphic to Sp(2, $\Re$) for which the U(2)-boson group is a familiar contraction limit (see also § 5). Consequently, SO(3, 2) ~ Sp(2, $\Re$) are also dynamical groups for the symmetrical representations of SU(3).

An $Sp(2, \mathfrak{R})$ unirrep can be labelled by the $U(1)\otimes SU(2)$ quantum numbers $\langle\sigma(\lambda)\rangle$ of its lowest weight state. For example, the positive parity states of the simple two-dimensional harmonic oscillator, shown in figure 1, carry the unirrep $\langle 1/2(0)\rangle$, where $2\sigma = 1$ denotes the harmonic oscillator energy of the lowest weight state in harmonic oscillator units. One observes that, for this representation, every integer value of L occurs precisely once. Thus the $\langle 1/2(0)\rangle$ representation of $Sp(2, \mathfrak{R})$ is a model for SO(3) (see also § 5 and Bhaumik *et al* (1975)). However, it is not a direct sum of SU(3) unirreps. Nevertheless, every symmetrical SU(3) unirrep can be embedded in the $\langle 1/2(0)\rangle$ $Sp(2, \mathfrak{R})$ space (Le Blanc 1985). For example, the carrier space for the SU(3) representation $\{2\}$ is simply the direct sum of the $L = 0$ and 2 subspaces. In this way (as we shall show elsewhere), it is possible to obtain a boson expansion for SU(3), regain the $SU(3) \supset SO(3)$ reduced matrix elements given above and obtain an explicit realisation of the contraction of SU(3) to the rotor algebra, $[\mathfrak{R}^5]SO(3)$.

	—	
$\{6\}$	—	3
$\{4\}$	—	2
$\{2\}$	—	1
$\{0\}$	—	0
$\{n_1 n_2\}$	$\langle 1/2(0)\rangle$	L
U(2)	$Sp(2, \mathfrak{R})$	SO(3)

Figure 1. The U(2) and SO(3) contents of the $Sp(2, \mathfrak{R})$ unirrep $\langle 1/2(0)\rangle$.

The $U(n)$ content of an arbitrary lowest weight $Sp(n, \mathfrak{R})$ unirrep has been given recently by Rowe *et al* (1985b). In particular, the U(2) content of an $Sp(2, \mathfrak{R})$ unirrep $\langle\sigma(0)\rangle$ for $\sigma \geqslant 2$ is shown in figure 2. One sees that the $\langle\sigma(0)\rangle$ representation for $\sigma \geqslant 2$ is indeed a model for the symmetrical SU(3) representations. The spectrum of states is, in fact, precisely the same as for the U(2)-boson algebra. Indeed, the $Sp(2, \mathfrak{R})$ algebra in the $\langle\sigma(0)\rangle$ representation contracts to the above realisation of the U(2)-boson algebra as $\sigma \to \infty$.

	—	
$\{6\}, \{4, 2\}$	—	$\{30\}$ 3,1
$\{4\}, \{2, 2\}$	—	$\{20\}$ 2, 0
$\{2\}$	—	$\{10\}$ 1
$\{0\}$	—	$\{00\}$ 0
$\{n_1 n_2\}$	$\langle\sigma(0)\rangle$	$\{h_1 0\}$ L
U(2)	$Sp(2, \mathfrak{R})$	$SU(3) \supset SO(3)$

Figure 2. The U(2) and SU(3) contents of the $Sp(2, \mathfrak{R})$ unirrep $\langle\sigma(0)\rangle$ for $\sigma \geqslant 2$.

3. The SO(6, 2) model for SU(3)

To generate two-rowed representations of SU(3) in a Bargmann space, one needs a minimum of two three-dimensional Bargmann vectors $\boldsymbol{g}_1$ and $\boldsymbol{g}_2$. These vectors will generate symmetrical representations $\{h00\ldots 0\}$ of a U(6) group for which the usual reduction

$$U(6) \downarrow \mathscr{U}(2)\otimes U(3) : \{h00\ldots 0\} \downarrow \sum_{h_1+h_2=h} (h_1 h_2) \times \{h_1 h_2 0\} \tag{3.1}$$

in terms of the complementary subgroups $\mathscr{U}(2)$ and U(3) is known to be multiplicity free. The Lie algebras for the complementary subgroups $\mathscr{U}(2)$ and U(3) of this reduction are given respectively (with summation over repeated indices) by

$$\mathscr{U}(2)\colon\ \{\mathscr{C}_{\alpha\beta}=g_{\alpha i}\partial/\partial g_{\beta i}\},\qquad \mathrm{U}(3)\colon\ \{C_{ij}=g_{\alpha i}\partial/\partial g_{\alpha j}\}. \tag{3.2}$$

Note that under these definitions the boson vacuum

$$\langle g|0\rangle=1 \tag{3.3}$$

spans respectively a {000} unirrep of U(3) and a (00) unirrep of $\mathscr{U}(2)$. Under U(3), both g_1 and g_2 span {100} unirreps and, as a consequence, the {10} SU(3) unirrep occurs twice. In general, an SU(3) unirrep $\{h_1h_2\}$ occurs with multiplicity equal to the dimensionality of the $\mathscr{U}(2)$ unirrep (h_1h_2). Also, if one restricts to the subspace of $\mathscr{U}(2)$ lowest weight states, one obtains an SU(3) model space. This model space was discussed in detail by the authors elsewhere. The tensor structure of the enveloping algebra of this $\mathscr{U}(2)\otimes\mathrm{U}(3)$ group product was also studied therein and a canonical basis of SU(3) tensors was given in terms of $\mathscr{U}(2)\otimes\mathrm{SU}(3)$ tensors classified by SU(3) operator patterns (Le Blanc 1985, Le Blanc and Rowe 1986). Although very useful for many practical purposes (we have succeeded in obtaining closed form expressions for specific classes of SU(3) Wigner coefficients), this model space has the disadvantage that it does not carry a unirrep of any known dynamical group with a simple action which is therefore an impediment to a full resolution of the multiplicity problem. Undoubtedly, a group could be found with a complicated action. It is nevertheless worthwhile to consider other model spaces.

We recall that for the symmetrical unirreps of U(6), the O(6) reduction

$$\mathrm{U}(6)\downarrow\mathrm{O}(6)\colon\ \{h00\ldots0\}\downarrow[h00]+[h-2,0,0]+\ldots+[100]\ \text{or}\ [000] \tag{3.4}$$

is multiplicity free. Furthermore, SU(3) is known to be embedded in SO(6) and the reduction

$$\mathrm{SO}(6)\downarrow\mathscr{SO}(2)\otimes\mathrm{SU}(3)\colon\ [\lambda00]\downarrow[\nu]\otimes\{\lambda,(\lambda-\nu)/2\},$$
$$\nu=-\lambda,-\lambda+2,\ldots,\lambda-2,\lambda, \tag{3.5}$$

is again multiplicity free (Dragt 1965). In this reduction, the Lie algebras of the complementary subgroups SU(3) and $\mathscr{SO}(2)$ are respectively given by (Chacón *et al* 1984, Biedenharn and Flath 1984, Bracken and MacGibbon 1984)

$$\mathrm{SU}(3)\colon\ \left\{C_{ij}=g_{1i}\frac{\partial}{\partial g_{1j}}-g_{2j}\frac{\partial}{\partial g_{2i}}-\tfrac{1}{3}M\delta_{ij}\right\},$$
$$\mathscr{SO}(2)\colon\ \left\{M=g_{1i}\frac{\partial}{\partial g_{1i}}-g_{2i}\frac{\partial}{\partial g_{2i}}\right\}. \tag{3.6}$$

Under this realisation g_1 will still carry a {100} unirrep of U(3) but g_2 will now carry a $\{00-1\}$ unirrep of U(3). Thus g_2 will carry a {11} unirrep of SU(3). That SU(3) is embedded in SO(6) is easily verified with the following change of variables:

$$g_{1i}=-(1/\sqrt{2})(\eta_{1i}+\mathrm{i}\eta_{2i}),\qquad g_{2i}=(\mathrm{i}/\sqrt{2})(\eta_{1i}-\mathrm{i}\eta_{2i}). \tag{3.7}$$

In terms of the new variables η, an anti-Hermitian basis for the Lie algebra of SO(6) is given by

$$\mathrm{SO}(6)\colon\ \{X_{\mu\nu}=\eta_\mu\partial/\partial\eta_\nu-\eta_\nu\partial/\partial\eta_\mu\},\qquad \mu,\nu=(\alpha i),\ 1\leqslant\alpha\leqslant2,\ 1\leqslant i\leqslant3, \tag{3.8}$$

and we easily find that the Lie algebras of the complementary subgroups are now given by (cf Bracken and MacGibbon 1984, Biedenharn and Flath 1984, Hecht and Pang 1969)

$$\begin{aligned} &\mathrm{SU}(3)\colon\ \{C_{ij}=\tfrac{1}{2}(X_{(1i)(1j)}+X_{(2i)(2j)}+\mathrm{i}X_{(2i)(1j)}+\mathrm{i}X_{(2j)(1i)})-\tfrac{1}{3}M\delta_{ij}\},\\ &\mathcal{SO}(2)\colon\ \{M=-\mathrm{i}X_{(1i)(2i)}\}. \end{aligned} \tag{3.9}$$

From (3.4) and (3.5), it is clear that if we retain only the O(6) unirrep $[h]$ in each U(6) unirrep $\{h\}$, then the direct sum of these O(6) maximal unirreps contains every SU(3) unirrep precisely once. Thus it is an SU(3) model space. This space is known to carry a unirrep of the group SO(6, 2) (see § 5). This was the starting point of the analysis of Biedenharn and Flath (1984) and Bracken and MacGibbon (1984). Biedenharn and Flath (1984) proceeded to give the SU(3) decomposition of the enveloping algebra of SO(6, 2) which allowed them to study the tensor structure of SU(3) in terms of the fundamental Wigner operators of SU(3) in a coordinate-free way. Their results may be compared to those obtained by the authors elsewhere (Le Blanc and Rowe 1985c, 1986).

Unfortunately, the SO(6, 2) model as presented above has some undesirable features. First, although a non-unitary realisation can be given relatively easily for the so(6, 2) algebra in terms of Bargmann variables (Biedenharn and Flath 1984, Bracken and MacGibbon 1984), its unitary realisation is complicated by the appearance of non-polynomial expressions in these same variables. Then, the SU(3) group is not canonically embedded in SO(6) and the decomposition and classification of more general SU(3) tensors appearing in the enveloping algebra of SO(6, 2) is therefore obscured. Furthermore, the fundamental Wigner (shift) tensor operators carrying irreducible representations {10} and {11} of SU(3) have complicated expressions in this realisation (cf Biedenharn and Flath 1984, Bracken and MacGibbon 1984). It will be shown in § 4 that these problems can be remedied.

4. The SO*(8) model for SU(3)

Starting from the Lie algebra isomorphism $\mathrm{u}(4)\sim\mathrm{so}(6)\oplus\mathrm{so}(2)$, we construct a Lie algebra so*(8) isomorphic to so(6, 2) in which su(3) is canonically embedded. In § 5, we go on to show that the simple but apparently non-unitary realisation of so(6, 2) in terms of six Bargmann variables is nothing but a coherent state realisation of this algebra which is, in fact, unitary with respect to the coherent state measure but not with respect to the Bargmann measure.

The Lie algebra u(4) is given in terms of two four-dimensional Bargmann vectors $g_{\alpha\mu}$, $\alpha=1,2$, $\mu=1,\dots,4$ (with summation over repeated indices) by

$$C_{\mu\nu}=\tfrac{1}{2}\left(g_{\alpha\mu}\frac{\partial}{\partial g_{\alpha\nu}}+\frac{\partial}{\partial g_{\alpha\nu}}g_{\alpha\mu}\right)=g_{\alpha\mu}\partial/\partial g_{\alpha\nu}+\delta_{\mu\nu},\qquad \mu,\nu=1,4, \tag{4.1}$$

with the u(3) subalgebra given by the restriction of the indices μ, ν to $i,j=1,2,3$. According to this definition, the boson vacuum

$$\langle g|0\rangle=1$$

carries a unirrep $1\{000\}\equiv\{1111\}$ of U(4) where we use the notation $\{h_1h_2h_3h_4\}\equiv h_4\{h_1-h_4, h_2-h_4, h_3-h_4\}$.

We now augment this u(4) algebra by the addition of six Cartan raising operators, antisymmetrical in their indices,

$$A_{\mu\nu} = -A_{\nu\mu} = \begin{vmatrix} g_{1\mu} & g_{1\nu} \\ g_{2\mu} & g_{2\nu} \end{vmatrix}, \tag{4.2}$$

and six Cartan lowering operators, also antisymmetrical in their indices,

$$B_{\mu\nu} = -B_{\nu\mu} = A^{\dagger}_{\mu\nu} = \begin{vmatrix} \partial/\partial g_{1\mu} & \partial/\partial g_{1\nu} \\ \partial/\partial g_{2\mu} & \partial/\partial g_{2\nu} \end{vmatrix}. \tag{4.3}$$

We find the following commutation relations:

$$\begin{aligned} &[C_{\mu\nu}, A_{\gamma\delta}] = \delta_{\nu\gamma}A_{\mu\delta} + \delta_{\nu\delta}A_{\gamma\mu}, \qquad [C_{\mu\nu}, B_{\gamma\delta}] = -\delta_{\mu\gamma}B_{\nu\delta} - \delta_{\mu\delta}B_{\gamma\nu}, \\ &[B_{\mu\nu}, A_{\gamma\delta}] = \delta_{\delta\nu}C_{\gamma\mu} + \delta_{\gamma\mu}C_{\delta\nu} - \delta_{\nu\gamma}C_{\delta\mu} - \delta_{\mu\delta}C_{\gamma\nu}. \end{aligned} \tag{4.4}$$

Under u(4), A is a $\{1100\}$ tensor, B is a $\{00-1-1\}$ tensor while the u(4) subalgebra is spanned by the components of a $\{100-1\}$ tensor.

The Lie algebra (4.4) is highly reminiscent of the Lie algebra associated with the non-compact symplectic group Sp(4, $\Re$) (see e.g. Rowe *et al* 1985b). In fact, the main difference between the two algebras resides in the fact that the symplectic raising and lowering operators $A^{\mathrm{sp}}_{\mu\nu}$ and $B^{\mathrm{sp}}_{\mu\nu}$ are symmetrical in their indices while here $A_{\mu\nu}$ and $B_{\mu\nu}$ are antisymmetrical. The algebra (4.4) can duly be generalised to any dimension n. The corresponding groups have already been identified. They are associated with a particular real form of the D_n class of Lie groups and are denoted $\mathrm{SO}^*(2n)$ by Gilmore (1974) who also noted the isomorphism $\mathrm{SO}(6,2) \simeq \mathrm{SO}^*(8)$. Note that the index n stands for the index of their maximal compact subgroups U(n).

We consider the unitary irreducible representations of the positive discrete series of $\mathrm{SO}^*(2n)$ that are infinite-dimensional lowest weight unirreps as for Sp($n, \Re$)(Rowe *et al* 1985b), i.e. they are characterised by a lowest weight state for which

$$\begin{aligned} &B_{\mu\nu}|\{h\}\mathrm{LW}\rangle = 0, \\ &C_{\mu\nu}|\{h\}\mathrm{LW}\rangle = 0, \qquad \mu \leqslant \nu, \\ &C_{\mu\mu}|\{h\}\mathrm{LW}\rangle = h_\mu|\{h\}\mathrm{LW}\rangle, \end{aligned} \tag{4.5}$$

where h is the partition characterising the U(n) unirrep generated from the lowest weight state by the u(n) subalgebra. Evidently, the realisation of $\mathrm{SO}^*(8)$ given by (4.1)-(4.3) corresponds to the representation $1\{000\} \equiv \{1111\}$.

Now the set of polynomials of degree h_1 in the Cartan raising operators A of $\mathrm{SO}^*(8)$ will span the U(4) unirrep $\{h_1h_100\}$. Furthermore, under the $\mathrm{U}(4) \downarrow \mathrm{U}(3)$ reduction, we retrieve from the well known betweenness conditions of the Gel'fand patterns the set of SU(3) unirreps:

$$\mathrm{SU}(4) \downarrow \mathrm{SU}(3)\colon\ \{h_1h_10\} \downarrow \sum_{h_2=0}^{h_1} \{h_1h_2\}.$$

For example, the SU(3) lowest weight state belonging to the unirrep $\{h_1h_2\}$ will be given in our Bargmann space by

$$\langle g|\{h_1h_2\}\mathrm{LW}\rangle = M(h_1h_2)A_{14}^{h_1-h_2}(g)A_{12}^{h_2}(g) \tag{4.6}$$

where $M(h_1h_2)$ is a normalisation constant. Thus it is apparent that the $1\{000\}$ unirrep of $\mathrm{SO}^*(8)$ is an SU(3) model.

This SU(3) model has some very useful properties. In particular, as a consequence of the fact that the $u(3) \supset su(3)$ subalgebras are canonically embedded in $so^*(8)$, the $so^*(8)$ algebra naturally decomposes into irreducible SU(3) tensors which are in fact Wigner tensors (see (4.9)).

A fundamental U(3) Wigner tensor, using Biedenharn and Flath notation (1984), is characterised by a set of shifts

$$\Delta = (\Delta_1, \Delta_2, \Delta_3) \tag{4.7}$$

by which we mean that this tensor will map an SU(3) unirrep $\{h_1 h_2\}$ of our model space to a new unirrep labelled by

$$\{h_1' h_2'\} = \{h_1 + \Delta_1 - \Delta_3,\ h_2 + \Delta_2 - \Delta_3\}. \tag{4.8}$$

We easily verify that the following SU(3) vector tensors in the $so^*(8)$ algebra have the shift properties:

$$\begin{aligned} &\tfrac{1}{2}\varepsilon_{ijk}A_{jk}\colon\ \Delta = (110), && A_{i4}\colon\ \Delta = (100), \\ &C_{i4}\colon\ \Delta = (010), && C_{4i}\colon\ \Delta = (101), \\ &B_{i4}\colon\ \Delta(011), && \tfrac{1}{2}\varepsilon_{ijk}B_{jk}\colon\ \Delta = (001). \end{aligned} \tag{4.9}$$

Therefore, the expressions for the fundamental Wigner operators are seen to be extremely simple in this model. Their Hermiticity relations are also clearly apparent.

We could now proceed to study the tensor structure of SU(3) within the SO*(8) framework but refrain from doing so. We nevertheless note that, although the analysis of Biedenharn and Flath is coordinate free and therefore independent of the realisation of the model, it can be shown (Le Blanc and Rowe 1985d) that a classification of all SU(3) tensors belonging to the enveloping algebra of SO*(8) yields somewhat different results. As u(4) is the maximal compact subalgebra of $so^*(8)$, tensors in the enveloping algebra are first classified as U(4) tensors and, due to the multiplicity-free reduction $U(4) \downarrow U(3)$, such a classification resolves some of the ambiguities of the construction given by Biedenharn and Flath. Furthermore, while most of their proofs are constructive, Le Blanc and Rowe's are deductive (therefore more transparent) and only use two properties of the U(4) tensors, namely their SU(4) weight properties and their straightforward $U(4) \downarrow U(3)$ decomposition using the betweenness conditions of the corresponding Gel'fand patterns.

We conclude this section by giving the $so(6,2) \sim so^*(8)$ isomorphism. Recall that the Lie algebra for SO(6, 2) is given in terms of the 28 Hermitian generators J_{AB}, $A, B = 1, 8$, $J_{AB} = J^\dagger_{AB} = -J_{BA}$ by

$$[J_{AB}, J_{CD}] = \mathrm{i}(g_{AC}J_{BD} + g_{BD}J_{AC} - g_{AD}J_{BC} - g_{BC}J_{AD}), \tag{4.10}$$

where (g_{AB}) is the diagonal metric $g = (1, 1, 1, 1, 1, 1, -1, -1)$.

The isomorphism between so(6) and su(4) is immediate if one uses Wong's $so(n)$ lowering and raising operators (see Wong 1967, Hecht and Pang 1969). We then find the following correspondences for the Cartan subalgebra:

$$\begin{aligned} &C_{11} = \tfrac{1}{2}(J_{12} + J_{34} + J_{56} + J_{78}), && C_{22} = \tfrac{1}{2}(J_{12} - J_{34} - J_{56} + J_{78}), \\ &C_{33} = \tfrac{1}{2}(-J_{12} - J_{34} + J_{56} + J_{78}), && C_{44} = \tfrac{1}{2}(-J_{12} + J_{34} - J_{56} + J_{78}). \end{aligned} \tag{4.11}$$

The u(4) lowering operators are given by

$$C_{31}=\tfrac{1}{2}(J_{13}-\mathrm{i}J_{23}-\mathrm{i}J_{14}-J_{24}), \qquad C_{41}=\tfrac{1}{2}(J_{15}-\mathrm{i}J_{25}-\mathrm{i}J_{16}-J_{26}),$$
$$C_{21}=\tfrac{1}{2}(J_{45}-\mathrm{i}J_{46}+\mathrm{i}J_{35}+J_{36}), \qquad C_{42}=\tfrac{1}{2}(J_{13}-\mathrm{i}J_{23}+\mathrm{i}J_{14}+J_{24}), \quad (4.12)$$
$$C_{32}=\tfrac{1}{2}(-J_{15}+\mathrm{i}J_{25}-\mathrm{i}J_{16}-J_{26}), \qquad C_{43}=\tfrac{1}{2}(J_{45}-\mathrm{i}J_{46}-\mathrm{i}J_{35}-J_{36}),$$

and their Hermitian adjoints give the raising operators. The non-compact raising operators are given by

$$A_{12}=\tfrac{1}{2}(J_{17}+\mathrm{i}J_{27}-\mathrm{i}J_{18}+J_{28}), \qquad A_{13}=-\tfrac{1}{2}\mathrm{i}(J_{57}+\mathrm{i}J_{67}-\mathrm{i}J_{58}+J_{68}),$$
$$A_{14}=\tfrac{1}{2}\mathrm{i}(J_{37}+\mathrm{i}J_{47}-\mathrm{i}J_{38}+J_{48}), \qquad A_{23}=-\tfrac{1}{2}\mathrm{i}(J_{37}-\mathrm{i}J_{47}-\mathrm{i}J_{38}-J_{48}), \quad (4.13)$$
$$A_{24}=-\tfrac{1}{2}\mathrm{i}(J_{57}-\mathrm{i}J_{67}-\mathrm{i}J_{58}-J_{68}), \qquad A_{34}=\tfrac{1}{2}(J_{17}-\mathrm{i}J_{27}-\mathrm{i}J_{18}-J_{28}),$$

while the non-compact lowering operators $B_{\mu\nu}$ are given by

$$B_{\mu\nu}=A^{\dagger}_{\mu\nu}. \qquad (4.14)$$

The verification of the isomorphism between the so(6, 2) Lie algebra as given by (4.10) and the Lie algebra generated by $\{A, B, C\}$ (equation (4.4)) is then immediate if one uses the correspondences (4.11)-(4.14).

5. Coherent state representation for the discrete series of SO(n, 2)

In this section, we show that the non-unitary boson realisation of the so(6, 2) Lie algebra given by Biedenharn and Flath (1984) and Bracken and MacGibbon (1984) for the SO(6, 2) model is but a coherent state realisation. The coherent state realisation given below is general and applies to all discrete series of SO(n, 2).

In terms of the $(n+2)(n+1)/2$ Hermitian generators

$$J_{AB}=J^{\dagger}_{AB}=-J_{BA}, \qquad A, B=1,\ldots,n+2,$$

the Lie algebra for so(n, 2) is given by

$$[J_{AB}, J_{CD}]=\mathrm{i}(g_{AC}J_{BD}+g_{BD}J_{AC}-g_{AD}J_{BC}-g_{BC}J_{AD}), \qquad (5.1)$$

where (g_{AB}) is the $(n+2)\times(n+2)$ diagonal metric $(1, 1, \ldots, 1, -1, -1)$.

Before introducing the coherent state representation for the so(n, 2) algebra, we must express it in a Cartan basis. We therefore define

$$Z_\alpha=(1/\sqrt{2})(J_{\alpha,n+1}-\mathrm{i}J_{\alpha,n+2}), \qquad \alpha=1,\ldots,n,$$
$$D_\alpha=Z^{\dagger}_\alpha=(1/\sqrt{2})(J_{\alpha,n+1}+\mathrm{i}J_{\alpha,n+2}), \qquad \alpha=1,\ldots,n,$$
$$X_{\alpha\beta}=-\mathrm{i}J_{\alpha\beta}, \qquad \alpha,\beta=1,\ldots,n, \qquad (5.2)$$
$$N=J_{n+1,n+2}.$$

From (5.1) and (5.2), we then obtain (see also Bargmann and Todorov 1977)

$$[X_{\mu\nu}, Z_\alpha]=\delta_{\mu\alpha}Z_\nu-\delta_{\nu\alpha}Z_\mu,$$
$$[X_{\mu\nu}, D_\alpha]=[X_{\mu\nu}, Z_\alpha]^{\dagger}=\delta_{\mu\alpha}D_\nu-\delta_{\nu\alpha}D_\mu, \qquad (5.3a)$$
$$[D_\mu, Z_\nu]=X_{\mu\nu}+N\delta_{\mu\nu},$$

and

$$[N, A_\nu]=A_\nu, \qquad [N, B_\nu]=-B_\nu, \qquad [N, X_{\mu\nu}]=0. \qquad (5.3b)$$

In this basis, Z and D behave respectively as Cartan (n-vector) raising and lowering operators. The generator N spans an so(2) ~ u(1) subalgebra while the anti-Hermitian generators $(X_{\mu\nu})$ span the so(n) subalgebra. Together N and $(X_{\mu\nu})$ span the maximal compact subalgebra so(2)⊗so(n).

Generic discrete series representations of SO(n, 2) are identified by their lowest weight states, i.e. states of lowest weight with respect to the so(n) subalgebra that are also annihilated by the lowering operators D_μ,

$$D_\mu|n_o[m]; \text{LW}\rangle = 0, \qquad \forall\mu. \tag{5.4}$$

We will label such representations by

$$n_o[m_1 m_2 \ldots m_\nu], \qquad n = 2\nu \text{ or } 2\nu+1,$$

where n_o is the value of the so(2) number operator N on the lowest weight state and $[m_1 m_2 \ldots m_\nu]$ is the character of the SO(n) unirrep generated from the lowest weight state by the so(n) subalgebra.

Vector valued coherent state wavefunctions are defined by (Rowe 1984, Rowe *et al* 1985a)

$$|\psi(z)\rangle = \sum_\eta |\eta\rangle\langle\eta|e^{z\cdot D}|\psi\rangle \tag{5.5}$$

where $z\cdot D = z_\alpha D_\alpha$ and where $\{|\eta\rangle\}$ spans the SO(n) unirrep based on the SO(n, 2) lowest weight state. We shall refer to this unirrep as the 'intrinsic' SO(n) unirrep. The wavefunctions (5.5) are seen to be holomorphic functions of the n-dimensional Bargmann vector z_μ which expresses the factor space

$$\text{SO}(n,2)/(\text{SO}(n)\otimes\text{SO}(2))$$

in a coordinate system.

The coherent state representation $\Gamma(\mathcal{O})$ for any operator $\mathcal{O}$ belonging to the so(n, 2) Lie algebra is defined by

$$\begin{aligned}\Gamma(\mathcal{O})|\psi(z)\rangle &= \sum_\eta |\eta\rangle\langle\eta|e^{z\cdot D}\mathcal{O}|\psi\rangle \\ &= \sum_\eta |\eta\rangle\langle\eta|\{\mathcal{O}+[z\cdot D,\mathcal{O}]+\tfrac{1}{2}[z\cdot D,[z\cdot D,\mathcal{O}]]+\ldots\}e^{z\cdot D}|\psi\rangle.\end{aligned} \tag{5.6}$$

Using (5.3) and (5.6), we therefore obtain the following coherent state realisation Γ for SO(n, 2):

$$\begin{aligned}&\Gamma(Z_\nu) = z_\alpha X^i_{\alpha\nu} + (N^i + z\cdot\nabla - 1)z_\nu - \tfrac{1}{2}z\cdot z\nabla_\nu,\\ &\Gamma(D_\nu) = \nabla_\nu, \qquad \Gamma(X_{\mu\nu}) = X^i_{\mu\nu} + (z_\nu\nabla_\mu - z_\mu\nabla_\nu),\\ &\Gamma(N) = N^i + z\cdot\nabla,\end{aligned} \tag{5.7}$$

with $\nabla_\nu = \partial/\partial z_\nu$. N^i and $X^i_{\mu\nu}$ span an intrinsic so(2)⊕so(n) algebra that commutes with the Weyl-Heisenberg (boson) algebra $\{z_\nu, \nabla_\mu, \delta_{\mu\nu}\}$. Equivalent expressions are also obtained in the partially coherent state formalism of Deenen and Quesne (1984).

For a representation $n_o[m] = n_o[0]$, for which the intrinsic SO(n) representation is the trivial identity representation, the coherent state realisation (5.7) simplifies to the purely boson realisation

$$\begin{aligned}&\Gamma_b(Z_\nu) = (n_o + z\cdot\nabla - 1)z_\nu - \tfrac{1}{2}z\cdot z\nabla_\nu,\\ &\Gamma_b(D_\nu) = \nabla_\nu, \qquad \Gamma_b(X_{\mu\nu}) = z_\nu\nabla_\mu - z_\mu\nabla_\nu,\\ &\Gamma_b(N) = n_o + z\cdot\nabla.\end{aligned} \tag{5.8}$$

The particular boson realisation Γ_o for which $n_o = (n/2-1)$ corresponds to the fundamental SO(n, 2) representation and has been given previously by Lohe and Hurst (1971) and Bargmann and Todorov (1977). The generators $\Gamma_o(Z)$ for this particular representation are often referred to as traceless bosons. They satisfy

$$[\nabla^2, \Gamma_o(Z_\nu)] = 2z_\nu\nabla^2 \tag{5.9}$$

and, as a consequence, the fundamental boson realisation of the so(n, 2) algebra generates the space of all harmonic polynomials (symmetrical unirreps $[\lambda 00\ldots0]$ of SO(n)) in an n-dimensional Bargmann space (Bargmann and Todorov 1977, Lohe and Hurst 1971), i.e. all polynomials $P^{(\lambda)}(z)$ of degree λ in z such that

$$\nabla^2 P^{(\lambda)}(z) = 0. \tag{5.10}$$

In general, a vector coherent state representation is carried by the direct product of a space of Bargmann polynomials in z and a space of vectors carrying a representation $[m]$ of the intrinsic SO(n) algebra.

The lowest weight state for a boson realisation Γ_b is the boson vacuum

$$\langle z|0\rangle = 1 \tag{5.11}$$

which is obviously an SO(n) scalar. Decomposition under SO(n) of the fundamental irreducible representation $(n/2-1)[00\ldots0]$ of SO(n, 2) is particularly simple in the light of the above remarks. We have

$$\mathrm{SO}(n,2)\downarrow\mathrm{SO}(n)\colon\ (n/2-1)[00\ldots0]\downarrow\sum_{\lambda=0}^{\infty}[\lambda 0\ldots0]. \tag{5.12}$$

The SO(6, 2) model for SU(3) as presented by Biedenharn and Flath (1984) and Bracken and MacGibbon (1984) thus corresponds to the unirrep 2[000] of SO(6, 2). Under the transformation (3.7), it is also easily verified that the Wigner operators given by Bracken and MacGibbon correspond to the above Γ_o representation for SO(6, 2).

We remark that $\Gamma(Z)$ and $\Gamma(D)$ are not Hermitian conjugates with respect to the Bargmann measure and the realisation (5.7) is therefore not unitary with respect to this measure. It is of course unitary, by construction, with respect to the appropriate coherent state measure (Bargmann and Todorov 1977, Deenen and Quesne 1984, Rowe 1984, Rowe *et al* 1985a). However, if one defines a Hermitian transformation $H = H^\dagger$ (Rowe 1984) such that

$$\Gamma(Z_\mu) = H^2 z_\mu H^{-2}, \tag{5.13}$$

then the realisation $\gamma = H^{-1}\Gamma H$ of so(n, 2) is unitary with respect to the Bargmann measure. Indeed,

$$\gamma(Z_\mu) = Hz_\mu H^{-1} \qquad \text{and} \qquad \gamma(D_\mu) = H^{-1}\nabla_\mu H \tag{5.14}$$

are then Hermitian conjugates while the commutation relations for the representation γ are still identical to the ones given by (5.3). Also, we may require H to be an SO(n) scalar so that

$$\begin{aligned}\gamma(X_{\mu\nu}) &= \Gamma(X_{\mu\nu}) = X^i_{\mu\nu} + z_\nu\nabla_\mu - z_\mu\nabla_\nu,\\ \gamma(N) &= \Gamma(N) = N^i + z\cdot\nabla.\end{aligned} \tag{5.15}$$

The solution of (5.13) for H and the calculation of SO(n, 2) reduced matrix elements is greatly facilitated by the technique of Rowe *et al* (1984) of expressing $\Gamma(Z_\mu)$ in the form

$$\Gamma(Z_\mu) = [\Lambda, z_\mu], \tag{5.16a}$$

where Λ is the SO(2)⊗SO(n) scalar

$$\Lambda=\tfrac{1}{2}[(2N^i+z\cdot\nabla-1)z\cdot\nabla-\tfrac{1}{2}z\cdot z\nabla^2+(z_\beta\nabla_\alpha-z_\alpha\nabla_\beta)X^i_{\beta\alpha}]. \tag{5.16b}$$

In particular, analytic solutions are possible whenever the SO(n, 2) ↓ SO(2)⊗SO(n) reduction is multiplicity free as it is for the $n_o[0]$ representations.

When $n_o\to\infty$, we find that the so(n, 2) algebra contracts to

$$\begin{aligned}&[\gamma(Z_\mu),\gamma(D_\nu)]\to n_o\delta_{\mu\nu},\\ &\gamma(Z_\mu)\to\sqrt{n_o}z_\mu, \qquad \gamma(D_\mu)\to\sqrt{n_o}\partial_\mu\end{aligned} \tag{5.17}$$

(cf Rosensteel and Rowe (1982, 1983) and Rowe (1984) for the corresponding contraction of sp(n, $\mathfrak{R}$)). Therefore, the basis $\{\gamma(N),\gamma(X_{\mu\nu}),\gamma(Z_\nu),\gamma(D_\nu)\}$ generates under contraction an algebra that we easily identify as being the Lie algebra of the group product

$$(\mathrm{SO}(2)\otimes\mathrm{SO}(n))[\mathrm{HW}(n)] \tag{5.18}$$

with bases given respectively by

$$\begin{aligned}&\mathrm{SO}(n)\colon\ \{X_{\mu\nu}\}, \qquad \mathrm{HW}(n)\colon\ \{z_\mu,\partial_\nu,\delta_{\mu\nu}\},\\ &\mathrm{SO}(2)\simeq\mathrm{U}(1)\colon\ \{N=z_\alpha\partial_\alpha\}\end{aligned} \tag{5.19}$$

(see § 2 for the $n=3$ case).

6. Coherent state representation for the discrete series of SO*(2n) and calculation in an SO*(8) model for SU(3)

Using techniques parallel to those used in § 5, we derive the vector coherent state realisation of the so*($2n$) algebras, duly generalised from equation (4.4),

$$\begin{aligned}&\Gamma(A_{\mu\nu})=(C^iz)_{\mu\nu}-(C^iz)_{\nu\mu}-(n-1)z_{\mu\nu}-(z\nabla z)_{\mu\nu},\\ &\Gamma(B_{\mu\nu})=\nabla_{\mu\nu}, \qquad \Gamma(C_{\mu\nu})=C^i_{\mu\nu}-(z\nabla)_{\mu\nu},\end{aligned} \tag{6.1}$$

where $\mu,\ \nu=1,\ldots,n$, $z_{\mu\nu}=-z_{\nu\mu}$ are antisymmetric Bargmann variables and $\nabla_{\mu\nu}=-\nabla_{\nu\mu}=\partial/\partial z_{\mu\nu}$. Thus $z_{\mu\nu}$ and $\nabla_{\mu\nu}$ are components of a Heisenberg-Weyl algebra, HW($n(n-1)/2$), satisfying

$$[\nabla_{\mu\nu},z_{\alpha\beta}]=\delta_{\mu\alpha}\delta_{\nu\beta}-\delta_{\mu\beta}\delta_{\nu\alpha}. \tag{6.2}$$

The $C^i_{\mu\nu}$ are the components of an intrinsic U(n) algebra that commutes with the Heisenberg-Weyl algebra (6.2) and we use the matrix notation for products, e.g.

$$(C^iz)_{\mu\nu}=C^i_{\mu\alpha}z_{\alpha\nu}.$$

If $\{|\{h\};\eta\rangle\}$ denotes a basis for a unirrep $\{h\}$ of the intrinsic u(n) algebra, then a basis for the coherent state representation (6.1) of so*($2n$) is given by a subset of the product states (Rowe *et al* 1985b, Le Blanc and Rowe 1985a)

$$\{(z\times z\times\ldots\times z)^{\{\delta\}}\otimes|\{h\}\rangle\}, \tag{6.3}$$

where $(z\times z\times\ldots\times z)^{\{\delta\}}$ is a U(n) coupled Bargmann polynomial in the variables $(z_{\mu\nu})$ which individually carry a unirrep {110 . . . 0} of U(n).

The realisation (6.1) is transformed to a realisation that is unitary with respect to the Bargmann measure by a Hermitian transformation $\Gamma \to \gamma = K^{-1}\Gamma K$ where K satisfies

$$\Gamma(A_{\mu\nu}) = K^2 z_{\mu\nu} K^{-2}. \tag{6.4}$$

The solution of this equation is greatly facilitated, following Rowe *et al* (1984), by expressing $\Gamma(A_{\mu\nu})$ in the form

$$\Gamma(A_{\mu\nu}) = [\Omega, z_{\mu\nu}] \tag{6.5}$$

where

$$\Omega = \tfrac{1}{4}\operatorname{Tr}(z\nabla z\tilde{\nabla}) + \tfrac{1}{4}(n-1)\operatorname{Tr}(z\nabla) - \operatorname{Tr}(z\nabla C^{i}). \tag{6.6}$$

Specialising to the unirrep $1\{000\}$ of SO*(8), for which $C^{i}_{\mu\nu} = \delta_{\mu\nu}$, we obtain

$$\begin{aligned} &\Gamma_{o}(A_{\mu\nu}) = [\Omega_{o}, z_{\mu\nu}] = -z_{\mu\nu} - (z\nabla z)_{\mu\nu}, \\ &\Gamma_{o}(B_{\mu\nu}) = \nabla_{\mu\nu}, \qquad \Gamma_{o}(C_{\mu\nu}) = \delta_{\mu\nu} - (z\nabla)_{\mu\nu}, \end{aligned} \tag{6.7}$$

with

$$\Omega_{o} = \tfrac{1}{4}\operatorname{Tr}(z\nabla z\nabla) - \tfrac{1}{4}\operatorname{Tr}(z\nabla). \tag{6.8}$$

We find that Ω_o has eigenvalues

$$\Omega_{o}(h_1) = (h_1+3)h_1/2 \tag{6.9}$$

on the states

$$\langle z|\{h_1 h_1 00\};\ \xi\rangle \tag{6.10}$$

which are Bargmann polynomials of degree h_1 in the z. The label ξ stands here for the usual three-rowed Gel'fand pattern appropriate for the canonical $\mathrm{U}(4) \supset \mathrm{U}(3) \supset \mathrm{U}(2) \supset \mathrm{U}(1)$ reduction.

Matrix elements for the generators of the so*(8) algebra are given by the product of an SU(4) Wigner coefficient times some SU(4) reduced matrix element since the so*(8) algebra is naturally expressed in terms of SU(4) tensors (equation (4.4)).

The reduced matrix element for a generator $C_{\mu\nu}$ of the su(4) subalgebra is proportional to the square root of the su(4) quadratic Casimir invariant, up to normalisation of this subalgebra.

Using techniques similar to Rowe (1984) and Rowe *et al* (1985b), we find that the reduced matrix element for the raising operator $A^{\{110\}}$ is equal to

$$\begin{aligned} &\langle\{h_1+1, h_1+1, 0\}\|A^{\{110\}}\|\{h_1 h_1 0\}\rangle \\ &\qquad = [\Omega_{o}(h_1+1) - \Omega_{o}(h_1)]^{1/2}(\{h_1+1, h_1+1, 0\}\|z\|\{h_1, h_1, 0\}) \\ &\qquad = [(h_1+2)(h_1+1)]^{1/2} \end{aligned} \tag{6.11}$$

where

$$(\{h_1+1, h_1+1, 0\}\|z\|\{h_1, h_1, 0\}) = (h_1+1)^{1/2} \tag{6.12}$$

is the stretched boson reduced matrix element for the Heisenberg–Weyl algebra (6.2). The reduced matrix element for the lowering operators B can be easily deduced from (6.11) using Hermiticity considerations.

When applied to the representation (4.1)-(4.3) of so*(8), the result (6.11) can be used to give

$$\langle g|\{h_1h_10\}; \text{LW}\rangle = \frac{1}{h_1!(h_1+1)^{1/2}} \begin{vmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{vmatrix}^{h_1}, \tag{6.13}$$

from which all other states of the SU(4) unirrep $\{h_1h_10\}$ can be reached using the SU(4) raising operators.

Therefore, the knowledge of a few SU(4) Wigner coefficients should tremendously simplify the computation of matrix elements of a complete set of tensors for SU(3) arising in the enveloping algebra of the fundamental representation of SO*(8) (Biedenharn and Flath 1984) and hence should offer a very powerful tool toward a concrete resolution of the tensor structure of SU(3) in the context of a dynamical group for SU(3), namely SO*(8).

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